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The Emergence of Hydrodynamic Equations from Quantum Theory: A Decoherent Histories Analysis

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The most general description of the classical world is in terms of local densities (such as number, momentum, energy), and these typically evolve according to evolution equations of hydrodynamic form. To explain the emergent classicality of these variables from an underlying quantum theory, it is therefore necessary to show, first, that these variables exhibit negligible interference, and second, that the probabilities for their histories are peaked around hydrodynamic evolution. The implementation of this program in the context of the decoherent histories approach to quantum theory is described. It is argued that, for a system of weakly interacting particles, the eigenstates of local densities (averaged over a sufficiently large volume) remain approximate eigenstates under time evolution. This is a consequence of their close connection with the corresponding exactly conserved (and so exactly decoherent) quantities. The subsequent derivation of hydrodynamic equations from decoherent histories is discussed.

If the universe is described at the most fundamental level by quantum theory, why is it so very nearly classical? There are very many aspects to the issue of emergent classicality (see, for example, ref. 1 for an overview), but crucial to most of them is the demonstration that certain types of quantum states of the system in question exhibit negligible interference. Initial superpositions of such states may therefore be effectively replaced by statistical mixtures. This, loosely speaking, is decoherence, and has principally been demonstrated for the situation in which there is a distinguished system, such as a particle, coupled to its surrounding environment [2, 3].

Most generally, decoherence typically comes about when the variables describing the entire system of interest naturally separate into "slow" and "fast," whether or not this separation corresponds to, respectively, system

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and environment.[2] If the system consists of a large collection of interacting identical particles, as in a fluid, for example, the natural set of slow variables are the local densities: energy, momentum, number, charge, etc. These variables, in fact, are also the variables which provide the most complete description of the *classical* state of a fluid at a macroscopic level.

The most general demonstration of emergent classicality therefore consists in showing that, for a large collection of interacting particles described microscopially by quantum theory, the local densities become effectively classical. Although one might argue that the system–environment mechanism might play a role, since the collection of particles is coupled to each other, decoherence comes about in these situations for a different reason: it is because the local densities are almost conserved if averaged over a sufficiently large volume [5]. Hence, the approximate noninterference of local densities is due to the fact that they are close to a set of exactly conserved quantities, and exactly conserved quantities obey superselection rules.

Intuitively appealing though this argument is, it is clearly a *quantitative* issue. The object of this paper is to show that, under certain reasonable conditions, local densities averaged over a sufficiently large volume are indeed approximately decoherent as a result of their close connection to exact conservation.

We will approach the question using the decoherent histories approach to quantum theory [5–8], which has proved particularly useful for discussing emergent classicality in a variety of contexts. The central object of interest is the decoherence functional,

$$\begin{aligned} D(\underline{\alpha}, \underline{\alpha}') = \mathrm{Tr}\big(&P_{\alpha_n} e^{-(i/\hbar)H(t_n - t_{n-1})} \cdots P_{\alpha_2} e^{-(i/\hbar)H(t_2 - t_1)} P_{\alpha_1} |\Psi\rangle \\ &\times \langle\Psi| P_{\alpha_1'} e^{(i/\hbar)H(t_2 - t_1)} P_{\alpha_2'} \cdots P_{\alpha_{n-1}'} e^{(i/\hbar)H(t_n - t_{n-1})}\big) \end{aligned} \tag{1}$$

The histories are characterized by the initial state $|\Psi\rangle$ and by the strings of projection operators P_α at times t_1 to t_n (and $\underline{\alpha}$ denotes the string of alternatives $\alpha_1 \cdots \alpha_n$). Intuitively, the decoherence functional is a measure of the interference between pairs of histories $\underline{\alpha}$, $\underline{\alpha}'$. When it is zero for $\underline{\alpha} \neq \underline{\alpha}'$, we say that the histories are decoherent and probabilities $p(\underline{\alpha}) = D(\underline{\alpha}, \underline{\alpha})$ obeying the usual probability sum rules may be assigned to them. One can then ask whether these probabilities are strongly peaked about trajectories obeying classical equations of motion. For the local densities, these equations will be hydrodynamic equations, and these and closely related aspects of emergent classicality have been pursued at greater length elsewhere [4, 9, 10].

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[2] See ref. 4 for a discussion of the conditions under which the total Hilbert space may be written as a tensor product of system and environment Hilbert spaces.

We consider the class of systems which are described at the microscopic level by a Hamiltonian of the form

$$H = \sum_j \left(\frac{\mathbf{p}_j^2}{2m} + \sum_{l>j} \phi(|\mathbf{q}_j - \mathbf{q}_l|) \right) \tag{2}$$

For definiteness, we will concentrate on the case of a weakly interacting dilute gas, making brief reference to a one-dimensional chain of oscillators, but it will be clear that the physical ideas are reasonably general. The local densities of interest are the number density $n(\mathbf{x})$, the momentum density $\mathbf{g}(\mathbf{x})$, and the energy density $h(\mathbf{x})$, defined by

$$n(\mathbf{x}) = \sum_j \delta(\mathbf{x} - \mathbf{q}_j) \tag{3}$$

$$\mathbf{g}(\mathbf{x}) = \sum_j \mathbf{p}_j \, \delta(\mathbf{x} - \mathbf{q}_j) \tag{4}$$

$$h(\mathbf{x}) = \sum_j \left(\frac{\mathbf{p}_j^2}{2m} + \sum_{l>j} \phi(|\mathbf{q}_j - \mathbf{q}_l|) \right) \delta(\mathbf{x} - \mathbf{q}_j) \tag{5}$$

(suitably ordered, in the quantum case). We are interested in local densities smeared over a volume V. The effect of this is to replace the delta functions with a window function, denoted δ_V, which is zero outside V and 1 inside. It is also useful to work with the Fourier transforms of the local densities, denoted $n(\mathbf{k})$, $g(\mathbf{k})$, $h(\mathbf{k})$. So, for example, the local number density at wavelength $\mathbf{k}$ is

$$n(\mathbf{k}) = \sum_j e^{i\mathbf{k}\cdot\mathbf{q}_j} \tag{6}$$

Exact conservation is obtained in the limit $k = |\mathbf{k}| \to 0$, or $V \to \infty$, in (3)–(5).

We would like to compute the decoherence functional for histories consisting of projections onto the operators (3)–(5). (The construction of the projectors is described in more detail in ref. 10). In the case of exact conservation, $k = 0$, we have exact decoherence simply because the projectors in Eq. (1) all commute with H and with each other [11]. Our main task is therefore to show that as k increases from zero, there is still a nontrivial regime in which decoherence is approximately maintained.

A significant result in this direction has been established already by Calzetta and Hu for the case of local temperature $T(x)$ obeying the diffusion equation [12]. They took their initial state to be close to the equilibrium state, and worked backward from the diffusion equation plus fluctuations to deduce the influence functional it must have arisen from, from which the degree of decoherence could be deduced. Here, by contrast, initial macroscopic superposition states are considered. A more detailed comparison of these two approaches is certainly of interest.

We begin by rewriting the exact conservation case in a simple way that makes its generalization to locally conserved quantities more apparent. Suppose the histories are projections onto some conserved quantity Q. Let the initial state be a superposition of eigenstates of Q,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(|a\rangle + |b\rangle) \tag{7}$$

where $\langle a|b\rangle = 0$ and

$$\hat{Q}|a\rangle = a|a\rangle, \qquad \hat{Q}|b\rangle = b|b\rangle \tag{8}$$

Since the P_αs are projections onto Q, P_α either annihilates or preserves $|a\rangle$ and $|b\rangle$. Take the case of a history with just two moments of time (the generalization to more times is trivial). The only nonzero off-diagonal terms of the decoherence functional are of the form

$$\begin{aligned} D(\underline{\alpha}, \underline{\alpha}') &= \tfrac{1}{2}\mathrm{Tr}(P_{\alpha 2} e^{-(i/\hbar)Ht}|a\rangle\langle b|e^{(i/\hbar)Ht}) \\ &= \tfrac{1}{2}\mathrm{Tr}(P_{\alpha 2}|a_t\rangle\langle b_t|) \end{aligned} \tag{9}$$

But Q is conserved, hence $[P_{\alpha 2}, H] = 0$ and

$$\begin{aligned} P_{\alpha 2}|a_t\rangle &= P_{\alpha 2} e^{-(i/\hbar)Ht}|a\rangle \\ &= e^{-(i/\hbar)Ht} P_{\alpha 2}|a\rangle = |a_t\rangle \end{aligned} \tag{10}$$

(or equals zero if α_2 does not correspond to a). It follows that

$$\begin{aligned} D(\underline{\alpha}, \underline{\alpha}') &= \tfrac{1}{2}\mathrm{Tr}(P_{\alpha 2}|a_t\rangle\langle b_t|) \\ &= \langle b_t|a_t\rangle = \langle b|a\rangle = 0 \end{aligned} \tag{11}$$

and therefore we have decoherence.

Now suppose that the operator Q is one of the local densities (3)–(5), so is no longer exactly conserved. The steps up to Eq. (9) still hold. But to go further, we need to know how the eigenstates of the local densities behave under time evolution. A reasonable supposition, which will be justified, is the following. Let us suppose that under time evolution, the eigenstates of Q remain approximate eigenstates. That is, we initially have (7), but under evolution to time t,

$$\hat{Q}|a_t\rangle \approx \langle Q\rangle|a_t\rangle \tag{12}$$

or, more precisely,

$$\frac{(\Delta Q)^2}{\langle Q\rangle^2} \ll 1 \tag{13}$$

i.e., the state remains strongly peaked in the variable Q under time evolution. The states are then approximate eigenstates of the projectors, so that in place of Eq. (10), we have the approximate result $P_{\alpha_2}|a_t\rangle \approx |a_t\rangle$ (or equals zero) as long as the width of the projection is much greater than the uncertainty $(\Delta Q)^2$. Hence Eq. (11) follows approximately, and we get approximate decoherence to the extent that the approximation (13) holds.

The key point is therefore the following: approximate decoherence is assured for histories of operators Q whose eigenstates have the property that they remain strongly peaked in Q under time evolution, as characterized by (13). To demonstrate decoherence of the local densities, therefore, we need only find their eigenstates, and show that they satisfy the localization property (13) under time evolution. (Note, incidently, that the above argument actually assures decoherence of *any* variables Q satisfying the localization property. The particular significance of the local densities is that they are continuous functions of the coarse graining scale k, so are guaranteed to satisfy the requisite property if k is sufficiently close to zero.)

Since the three operators (3)–(5) do not commute, exact simultaneous eigenstates cannot be found. However, there are approximate simultanous eigenstates. For weak interactions, they are products of N identical terms,

$$|\Psi\rangle = |\psi\rangle \otimes |\psi\rangle \otimes \cdots \otimes |\psi\rangle \tag{14}$$

and are approximate eigenstates of all three operators for large N. The proof of this statement involves considering, for the local number density, for example, the object $(\Delta n(\mathbf{x}))^2/\langle n(\mathbf{x})\rangle^2$, and showing that it goes like $1/N$ for large N (see ref. 10, for example). It is essentially the central limit theorem (see also ref. 13). For the number and momentum density it relies on the fact that they are sums of identical one-particle operators. For the local energy density, it additionally requires the smearing volume to be sufficiently large compared to some length scale indicated by the interactions. Some tuning of the state $|\psi\rangle$ can be carried out to ensure that (14) is an optimal approximate eigenstate of all the local densities, but this will not be done here. [Also, the passage to exact eigenstates of $n(\mathbf{k})$, $g(\mathbf{k})$, $h(\mathbf{k})$ as $k \to 0$ can be seen explicitly if the one-particle states $|\psi\rangle$ are taken to be one-particle momentum eigenstates.]

The question is now what happens to the eigenstates (14) of the local densities under time evolution by the Hamiltonian (2)? Consider first the trivial but enlightening case in which there no interactions. In this case, the time-evolved eigenstates $|a_t\rangle$ remain of the product form (14), so they are *still* approximate eigenstates of the local densities (but with a time-evolved eigenvalue) for the same reasons as above. Hence there is approximate decoherence.

Decoherence in the noninteracting case comes about for two reasons. First, it is due to the fact that a state of the form (14) will remain strongly peaked about the average values of the local densities, $n(\mathbf{x})$, $g(\mathbf{x})$, $h(\mathbf{x})$, under time evolution, and thus the state is essentially undisturbed by the projectors (as long as their widths are sufficiently large). The strong peaking follows from the assumption of large N and from the fact that the local density operators are sums of identical one-particle operators. Second, it is due to the almost trivial fact that the orthogonality of the two elements of the initial state is preserved by unitary evolution.

This second fact is important because the first one is not always sufficient to guarantee decoherence. Although the state remains strongly peaked about the average values of the local densities, these average values do not necessarily obey deterministic equations. In the case of histories characterized by number density only, for example, $\langle n(\mathbf{x})\rangle$ at time t is *not* uniquely determined by $\langle n(\mathbf{x})\rangle$ at the initial time [in the state (14)]. That is, in Eq. (9), $|a_t\rangle$ and $|b_t\rangle$ may in fact be peaked about *the same* value of number density, even though the initial values are different. The decoherence is therefore not in fact due to an approximate determinism (such as that used in the phase space histories of Omnès [7]). It is necessary only that the evolved states are essentially undisturbed by the projectors, and therefore that the two orthogonal components of the initial state are eventually overlapped at the final time, as in Eq. (11), to give zero.

The next and most important task is to show that the above story is in fact still true, with qualifications, in the presence of interactions. The complete description of N interacting particles is generally extremely involved, but we can make some progress by restricting attention to a sufficiently dilute gas of weakly interacting components, and then making two assumptions which are standard in kinetic theory and nonequilibrium statistical mechanics [e.g., 14]. It is notationally convenient in what follows to work with a Wigner function rather than quantum state. Hence associated with the full N-particle wave function is an N-particle Wigner function $W_N(\mathbf{p}_1, \mathbf{q}_1, \ldots, \mathbf{p}_N, \mathbf{q}_N)$. For a dilute, weakly interacting gas, it is reasonable to assume that three-particle correlations are negligible. This is our first assumption. It means that all the physics is contained in the one-and two-particle reduced Wigner functions $W_1(\mathbf{p}_1, \mathbf{q}_1)$ and $W_2(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2)$. All higher order reduced Wigner functions will reduce to products of these.

We again take as our initial state the product state (14) (which is still an approximate eigenstate in the interacting case), and let it evolve, so correlations will develop. The degree to which the particles become correlated is contained in the two-particle distribution W_2 of the evolved eigenstate. On general grounds, we expect that the interparticle correlations will only be important on some length scale L, and beyond that length scale, they will be uncorrelated. That is, we will assume that

$$W_2(\mathbf{p}_1, \mathbf{q}_1, \mathbf{p}_2, \mathbf{q}_2) \approx W_1(\mathbf{p}_1, \mathbf{q}_1) W_1(\mathbf{p}_2, \mathbf{q}_2) \tag{15}$$

for $|\mathbf{q}_2 - \mathbf{q}_1| > L$, and otherwise W_2 will have a form indicating nontrivial correlations. This is our second assumption. It is physically reasonable, and it is in fact a key assumption in the derivation of the Boltzmann equation [14].

Note that the assumption (15) would not necessarily be appropriate for all possible initial quantum states. One could construct initial quantum states which would possess or develop nontrivial long-range correlations, for which this assumption may never hold. However, as seen in the noninteracting case, to demonstrate decoherence we only need to consider the time evolution of the special class of initial states which are eigenstates of the local densities. These particular initial states, which are of the form (14), approximately, do not have long-range correlations. It is therefore very plausible, at least for a dilute, weakly interacting gas, that they will develop only limited correlations under time evolution and the assumption (15) will hold.

Given the above assumptions, it is now reasonably straighforward to argue that the state is still strongly peaked about the average values of the local densities, as long as $V \gg L^3$. For example, for the number density, we have

$$\langle n(\mathbf{x}) \rangle = \sum_j \langle \delta_V(\mathbf{q}_j - \mathbf{x}) \rangle = N \int_V d^3\mathbf{q}\, p(\mathbf{q}) \tag{16}$$

where $p(\mathbf{q})$ is the one-particle probability distribution of $\mathbf{q}$ (obtained by integrating the one-particle Wigner function over $\mathbf{p}$). Similarly,

$$\begin{aligned} \langle n^2(\mathbf{x}) \rangle &= \sum_{jl} \langle \delta_V(\mathbf{q}_j - \mathbf{x}) \delta_V(\mathbf{q}_l - \mathbf{x}) \rangle \\ &= N \langle \delta_V \rangle + (N^2 - N) \langle \delta_V(\mathbf{q}_1 - x) \delta_V(\mathbf{q}_2 - x) \rangle \end{aligned} \tag{17}$$

where we have used $\delta_V^2 = \delta_V$, and also an assumption of identical particles to reduce the sum over j, l to particles labeled 1 and 2. We now have

$$\begin{aligned} (\Delta n(\mathbf{x}))^2 &= \langle n^2(\mathbf{x}) \rangle - \langle n(\mathbf{x}) \rangle^2 \\ &= N^2 (\langle \delta_V(\mathbf{q}_1 - \mathbf{x}) \delta_V(\mathbf{q}_2 - \mathbf{x}) \rangle - \langle \delta_V \rangle^2) \\ &\quad + N (\langle \delta_V \rangle - \langle \delta_V(\mathbf{q}_1 - \mathbf{x}) \delta_V(\mathbf{q}_2 - \mathbf{x}) \rangle) \end{aligned} \tag{18}$$

If there is no correlation at all between the particles, the coefficient of N^2 would vanish, so $(\Delta n(\mathbf{x}))^2 / \langle n(\mathbf{x}) \rangle^2$ would go like $1/N$, which goes to zero as $N \to \infty$. This is the standard central limit theorem result indicated earlier for the noninteracting case. With interactions, the coefficient of N^2 is no longer zero. We now need to show, therefore, that this term is still sufficiently small for $(\Delta n(\mathbf{x}))^2/\langle n(\mathbf{x})\rangle^2$ to remain small as $N \rightarrow \infty$. Introducing the two-particle distribution $p(\mathbf{q}_1, \mathbf{q}_2)$ (obtained by integrating $\mathbf{p}_1$, $\mathbf{p}_2$ out of W_2), we can readily show that the leftover terms as $N \rightarrow \infty$ are

$$\frac{(\Delta n(\mathbf{x}))^2}{\langle n(\mathbf{x})\rangle^2} = \frac{\int_V d^3\mathbf{q}_1 \int_V d^3\mathbf{q}_2 \, (p(\mathbf{q}_1, \mathbf{q}_2) - p(\mathbf{q}_1)p(\mathbf{q}_2))}{\left(\int_V d^3\mathbf{q} \, p(\mathbf{q})\right)^2} \tag{19}$$

This is clearly zero if there are no correlations. In the interacting case we use the assumption (15), which implies that

$$p(\mathbf{q}_1, \mathbf{q}_2) \approx p(\mathbf{q}_1)p(\mathbf{q}_2) \tag{20}$$

for $|\mathbf{q}_1 - \mathbf{q}_2| > L$, and otherwise nontrivial correlations exist. Hence the integral in the numerator takes contributions only from the region $|\mathbf{q}_1 - \mathbf{q}_2| < L$.

To see that (19) is small, note that in the numerator, the integral is over a volume V^2 in the six-dimensional two-particle configuration space. If $V \ll L^3$, the factorization of $p(\mathbf{q}_1, \mathbf{q}_2)$ for $|\mathbf{q}_1 - \mathbf{q}_2| > L$ makes no difference, since $\mathbf{q}_1$ and $\mathbf{q}_2$ can never be far enough apart in the integrand (assuming V is regular in shape). However, if $V \gg L^3$, the V^2-sized integration region is substantially reduced in size to $V \times L^3$. On dimensional grounds the numerator is therefore proportional to a number of order VL^3 and the denominator to V^2 (perhaps with other factors common to both). This means that

$$\frac{(\Delta n(\mathbf{x}))^2}{(n(\mathbf{x}))^2} \sim \frac{L^2}{V} \tag{21}$$

This order-of-magnitude estimate becomes exact if we assume that the probabilities are constant in the region of nontrivial correlation (another common assumption of kinetic theory [14]). Hence the state will be strongly peaked about the average of $n(\mathbf{x})$ if $V \gg L^3$.

In the one-dimensional oscillator chain model considered in ref. 15, the uncertainty in $n(k)$ [the one-dimensional version of Eq. (6)] can be computed explicitly in the special case of a Gaussian state. It is

$$(\Delta n(k))^2 = \sum_{j=1}^{N} \sum_{l=1}^{N} \langle e^{ikq_j}\rangle\langle e^{-ikq_l}\rangle(e^{k^2\sigma(q_j,q_l)} - 1) \tag{22}$$

where $\sigma(q_j, q_l) = \langle q_j q_l\rangle - \langle q_j\rangle\langle q_l\rangle$ measures the degree of correlation between different particles in the chain. As k increases from zero, the leading order terms in (22) are of the form $(\Delta n(k))^2 = k^2(\Delta X)^2$, where $X = \Sigma_j q_j$ (the center-of-mass coordinate), and since $\langle n(k)\rangle \sim N$, we have

$$\frac{(\Delta n(k))^2}{|\langle n(k)\rangle|^2} \sim \frac{k^2(\Delta X)^2}{N^2} \tag{23}$$

This will be very small as long as k^{-1} is much larger than the length scale of a single particle. $[\Delta n(k)]^2$ starts to grow very rapidly with k, and (23) is no longer valid when k^{-1} becomes less than the correlation length indicated by $\sigma(q_j, q_l)$. Hence the state is strongly peaked about the mean as long as the coarse-graining length scale k^{-1} remains much greater than the correlation length of the time-evolved local density eigenstates. This correlation length is considered in ref. 15 and found to be generally very small compared to the system size. A simple field theory model is also considered in ref. 15, confirming many of the expected features outlined here.

It is possible to see on physical grounds why one expects a result of the form (21) to hold quite generally. In the noninteracting case we used the central limit theorem result that $(\Delta n)^2/\langle n\rangle^2$ goes like $1/N$. In the interacting case, the state is no longer of the product form (14), but an analagous result still holds. The point is that the correlations that develop extend only over a (typically small) volume of size L^3, so the system breaks up into a large number of essentially identical uncorrelated regions of this size. Therefore each smearing volume V, if much greater than L^3, contains of order V/L^3 identical uncorrelated regions each of which contributes equally to the local density averaged over V. Loosely speaking, a central limit theorem-type result again applies, not to the N uncorrelated particles in the same state, but to the V/L^3 uncorrelated regions. So $1/N$ is replaced by L^3/V in the central limit theorem, and hence the above result.

Similar results hold for the local momentum and energy density. We have therefore demonstrated the desired result: eigenstates of the coarse-grained local densities remain approximate eigenstates under time evolution as long as the smearing volume is much greater than the correlation volume of these states. Decoherence of these variables then follows.

We now briefly consider the probabilities for histories. They are strongly peaked at each moment of time about the average values, $\langle n(\mathbf{x}, t)\rangle$, $\langle g(\mathbf{x}, t)\rangle$, $\langle h(\mathbf{x}, t)\rangle$, averaged in a local density eigenstate. The hydrodynamic equations (or even a closed set of equations) do not necessarily follow, however, since these require a local equilibrium initial state [16, 14], but we outline how this might come about.

The averages of the local densities depend only on the one-particle Wigner function (with a small correction depending on the two-particle function in the case of local energy density), hence the evolution equation of the local densities can be determined by obtaining an evolution equation for the one-particle Wigner function W_1. With a local density eigenstate as initial state, and with the two assumptions utilized above, we expect this evolution equation is the Boltzmann equation. We therefore evolve W_1, subject to the initial condition that it be equal to the one-particle Wigner function of a local density eigenstate. This initial Wigner function is not of local equilibrium form, but it is reasonable to expect that it will rapidly approach local equilibrium form under evolution according to the Boltzmann equation, and thereafter retain that form. [The temperature, chemical potential, etc., of the local equilibrium state will be determined by the average values $\langle h(\mathbf{x}, t)\rangle$, etc.] Hence, except for a short initial period during which the initial state settles down to local equilibrium form, the probabilities for histories will be peaked about hydrodynamic equations.

Some of these features can be seen in some detail in ref. 10, where the emergence of the diffusion equation was considered. The system studied was a collection of N foreign noninteracting particles in a background fluid. Decoherence was therefore provided largely by the fluid in this case, rather than by conservation, but the interest of the model is that it gives an explicit picture of the emergence of a hydrodynamic equation. Each foreign particle behaves like a quantum Brownian particle, whose evolution equation is well known. An initial state of the form (14) for the N foreign particles evolves into a mixed state of the form of an N-fold product,

$$\rho = \rho_1 \otimes \rho_1 \cdots \otimes \rho_1 \tag{24}$$

where each one-particle density operator ρ_1 describes quantum Brownian motion. From the Wigner function of ρ_1 it is readily shown that the one-particle position distribution obeys the diffusion equation at long times, from which it readily follows that the N-particle number density $n(x)$ also obeys the diffusion equation. This model is therefore a kind of "half-way house" between the decoherence-through-environment and decoherence-through-conservation models, but it helps to complete the general picture.

Summarizing, the final picture we have is therefore as follows. An initial state consisting of a superposition of local density eigenstates may be treated as a mixture of the same states, since they are decoherent. Each state separately will give probabilities peaked about hydrodynamic equations, with particular initial values of phenomenological parameters such as temperature, etc., and these will be different for each element of the mixture. We therefore have a statistical mixture of trajectories, each evolving according to hydrodynamic equations, but with different phenomenological parameters, i.e., to very different macroscopic states. More details of this work may be found elsewhere [15, 17].

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